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Holomorphic isometries of twistor spaces

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Abstract

Let $Z_g(M)$ be the twistor space over an oriented $2n$-dimensional Riemannian manifold (M, g) with nonpositive and parallel Ricci tensor. Let h and $\mathbb{J}$ be the natural metric and almost complex structure on $Z_g(M)$, respectively. We prove that any isometry of the twistor space $Z_g(M)$ preserves the horizontal and vertical distributions. When M is compact, we give an estimate of the dimension of the groups of isometries and holomorphic isometries on the twistor space. In particular, if M is a compact almost Kähler manifold with the same properties of curvature and the Ricci tensor is negative, then the group of the holomorphic isometries is finite.

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1. Introduction

The twistor space over an oriented $2n$-dimensional Riemannian manifold (M, g) is the bundle associated to the principal bundle of positive orthonormal frames on M, with standard fibre the Hermitian symmetric space $\mathrm{SO}(2n)/U(n)$. The fibres of this bundle parametrise the positive orthogonal complex structures of each tangent space to M. The Levi–Civita connection on the bundle of positive orthonormal frames on M gives a horizontal distribution on the twistor bundle and, consequently, a splitting of its tangent bundle. A natural metric h on the twistor bundle is defined by requiring that, the horizontal and vertical distributions are orthogonal, h restricted to the fibres is the standard metric on $\mathrm{SO}(2n)/U(n)$ and the bundle projection is a Riemannian submersion (see e.g. [4]). For a general discussion on these metrics see, for instance, [1], where such constructions are studied to produce examples of Einstein manifolds.

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One of the main features of the twistor space is the existence of an almost complex structure $\mathbb{J}$ on it, that is compatible with the natural metric h. This almost complex structure is integrable if and only if the base M is conformally flat, for $n > 2$, and anti-self-dual, for $n = 2$. Moreover, when M is compact, the metric h on $(Z_g(M), \mathbb{J})$ is Kählerian if and only if M is the sphere, for $n > 2$, and S^4 or the complex projective plane for $n = 2$ (see [2,3,7,11]).

The vanishing theorem of Bochner says that, if a compact Riemannian manifold has nonpositive Ricci tensor, then any Killing vector field is parallel. Moreover, if there exists a point such that the Ricci tensor is negative, then there are no nontrivial Killing vector fields and so the isometry group is finite. Similarly, if a compact almost Kähler manifold has nonpositive Ricci tensor, then any holomorphic vector field is parallel. Besides, if there exists a point such that the Ricci tensor is negative, then there are no nontrivial holomorphic vector fields and so the group of the biholomorphisms is finite (see [6,8,13]). A generalisation of Bochner's theorem for compact balanced Hermitian manifolds is given in [5].

In this paper we consider the twistor space over a Riemannian manifold M with nonpositive and parallel Ricci tensor. We prove that every isometry of the twistor space preserves the horizontal and vertical distributions. As a consequence, the Killing vector fields on the twistor space are characterised. In fact, if ξ is a such vector field, then ξ may be decomposed as a sum of a horizontal lift of a Killing vector field X on the base and a Killing vector field tangent to the fibres. Moreover, there are no nontrivial holomorphic vertical Killing vector fields on $Z_g(M)$. When M is compact, we give an estimate for the dimension of the groups of the isometries and holomorphic isometries of $Z_g(M)$. As a corollary, if the base M is compact almost Kähler with negative and parallel Ricci tensor, then there are no nontrivial holomorphic Killing vector fields on $(Z_g(M), h, \mathbb{J})$ and the holomorphic isometry group is finite.

Analogous results for the isometries of the tangent sphere bundle with the Sasaki metric are obtained in [9]. See also [14,15].

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2. Generalities

Let (M, g) be an oriented $2n$-dimensional Riemannian manifold, $\mathrm{SO}_g(M)$ be the $\mathrm{SO}(2n)$-principal bundle of oriented g-orthonormal frames on M and $\pi : \mathrm{SO}_g(M) \to M$ denotes the canonical projection. By definition, the *twistor space* $Z_g(M)$ over M is the associated bundle to $\mathrm{SO}_g(M)$ with standard fibre $Z(n) := \mathrm{SO}(2n)/U(n)$, i.e.

$$Z_g(M) = \mathrm{SO}_g(M) \times_{\mathrm{SO}(2n)} \mathrm{SO}(2n)/U(n),$$

where the group $\mathrm{SO}(2n)$ acts on the right on $\mathrm{SO}_g(M)$ and on the left on $\mathrm{SO}(2n)/U(n)$ in the following way:

$$\begin{aligned} &\mathrm{SO}(2n) \times \mathrm{SO}_g(M) \times \mathrm{SO}(2n)/U(n) \to \mathrm{SO}_g(M) \times \mathrm{SO}(2n)/U(n) \\ &(a, u, XU(n)) \mapsto (ua, a^{-1}XU(n)). \end{aligned}$$

We will denote by $r : Z_g(M) \to M$ and by $p : \mathrm{SO}_g(M) \to Z_g(M)$ the canonical projections.

Let ω be the Levi–Civita connection on $\mathrm{SO}_g(M)$ and, for any $u \in \mathrm{SO}_g(M)$,

$$T_u\mathrm{SO}_g(M) = H_u + V_u$$

the induced splitting of the tangent space $T_u\mathrm{SO}_g(M)$ with $H_u = \mathrm{Ker}\,\omega_u$.

The connection ω induces also a splitting of $TZ_g(M)$. For any $P \in Z_g(M)$, let $u \in p^{-1}(P)$; then

$$T_P Z_g(M) = p_*(H_u) + p_*(V_u),$$

where $\mathcal{H}_P := p_*(H_u)$ is the horizontal tangent space and $\mathcal{V}_P := p_*(V_u)$ is the tangent space to the fibre $Z_g(M)_{r(P)}$ of $Z_g(M)$ over $r(P)$, i. e. the vertical tangent space. We have

$$\mathcal{V}_P = \{X \in \mathfrak{so}(T_{r(P)}M, g_{r(P)}) | P \circ X = -X \circ P\},$$

where P is viewed as a positive orthogonal complex structure on $T_{r(P)}M$. We may identify the vertical tangent space at P to

$$\{X \in \mathfrak{so}(2n) | PX = -XP\} = \{\hat{Y}(P) | Y \in \mathfrak{so}(2n)\},$$

where $\hat{Y}(P) := [Y, P]$ is the Lie bracket on $\mathfrak{so}(2n)$.

For any pair $(\hat{A}, \hat{B})$ of vertical tangent vectors of $T_P Z_g(M)$, we set

$$\langle \hat{A}, \hat{B} \rangle_P := -\mathrm{Tr}(AB + APBP).$$

Then we define the *twistor metric* h on $Z_g(M)$ as

$$h(\xi, \eta) = g(r_*\xi^{\mathrm{H}}, r_*\eta^{\mathrm{H}}) + \langle \xi^{\mathrm{V}}, \eta^{\mathrm{V}} \rangle_P, \tag{1}$$

where $\xi^{\mathrm{H}}, \eta^{\mathrm{H}}$ denote the horizontal components of $\xi, \eta \in T_P Z_g(M)$ and $\xi^{\mathrm{V}}, \eta^{\mathrm{V}}$ the vertical ones. By definition, the horizontal and vertical distributions are h-orthogonal, and the projection $r : Z_g(M) \to M$ is a Riemannian submersion.

We recall also the definition of the almost complex structure $\mathbb{J}$ on $Z_g(M)$ (see for instance [4]). For any tangent vector X to $Z_g(M)$ in P, we set

$$\mathbb{J}_P(X) = (r_*^{-1} \circ P \circ r_*)(X^{\mathrm{H}}) + P \circ X^{\mathrm{V}}.$$

Then $\mathbb{J}$ defines an almost complex structure on $Z_g(M)$.

Now we are going to define local frames of horizontal and vertical distributions. Let $\{E_1, \ldots, E_{2n}\}$ be a local oriented g-orthonormal frame on M and Γ_{ij}^k the Christoffel's symbols. Then Γ_{ij}^k are skew-symmetric in the indices j and k, so that $\Gamma_{i\cdot}^{\cdot}$ is in $\mathfrak{so}(2n)$, for $i = 1, \ldots, 2n$. Let $\{\tilde{E}_1, \ldots, \tilde{E}_{2n}\}$ be the horizontal lifts of $\{E_1, \ldots, E_{2n}\}$ on the fibre bundle $\mathrm{SO}_g(M)$ and

$$\hat{E}_1 = p_*(\tilde{E}_1), \ldots, \hat{E}_{2n} = p_*(\tilde{E}_{2n}).$$

The local vector fields $\{\hat{E}_1, \dots, \hat{E}_{2n}\}$ give a local frame of the horizontal distribution on $Z_g(M)$. In terms of the trivialisation induced by $\{E_1, \dots, E_{2n}\}$, we have

$$\hat{E}_j(P) = E_j(r(P)) - [\Gamma^{\cdot}_{j\cdot}(r(P)), P]^r_s \frac{\partial}{\partial P^r_s}. \tag{2}$$

Let $\mathfrak{X}(\mathrm{SO}(2n)/U(n))$ be the Lie algebra of vector fields on $\mathrm{SO}(2n)/U(n)$. For any matrix $A \in \mathfrak{so}(2n)$, we denote by

$$\hat{}: \mathfrak{so}(2n) \to \mathfrak{X}(\mathrm{SO}(2n)/U(n))$$

the map defined by

$$\hat{}(A)(P) = [A, P]\frac{\partial}{\partial P}. \tag{3}$$

We recall that $\hat{}$ is an isomorphism for $n > 2$, while, for $n = 2$, its kernel is isomorphic to $\mathfrak{su}(2)$ (see [4,10]).

We define local vertical vector fields $\hat{A}_\alpha$, for $\alpha = 1, \dots, n^2 - n$, on $Z_g(M)$, by

$$\hat{A}_\alpha = [A_\alpha(r(P)), P]\frac{\partial}{\partial P}, \tag{4}$$

where $A_\alpha(\cdot)$ are in $\mathfrak{so}(2n)$. Note that, for $j = 1, \dots, 2n$,

$$\hat{E}_j(P) = E_j(r(P)) - \hat{\Gamma}^{\cdot}_{j\cdot}(P)$$

and

$$\mathbb{J}(\hat{E}_j(P)) = P^k_j \hat{E}_k(P).$$

Let us denote by $\hat{\nabla}$, $\check{\nabla}$ and ∇, the covariant derivative of $(Z_g(M), h)$, of the fibre $(Z_g(M))_x$ over any $x \in M$ and of (M, g), respectively.

We have the following formulas (see [4]):

$$[\hat{E}_i, \hat{E}_j] = (\Gamma^k_{ij} - \Gamma^k_{ji})\hat{E}_k - \hat{}(R^{\cdot}_{\cdot ij}), \tag{5}$$

$$[\hat{E}_i, \hat{A}_\alpha] = \hat{}(E_i(A_\alpha) + [\Gamma^{\cdot}_{i\cdot}, A_\alpha]), \tag{6}$$

$$[A_\alpha, A_\beta] = -\widehat{[A_\alpha, A_\beta]}, \tag{7}$$

$$\hat{\nabla}_{\hat{E}_i}\hat{E}_j = \Gamma^k_{ij}\hat{E}_k - \tfrac{1}{2}\hat{}(R^{\cdot}_{\cdot ij}), \tag{8}$$

$$\hat{\nabla}_{\hat{E}_i}\hat{A}_\alpha = \tfrac{1}{2}h(\hat{}(R^{\cdot}_{\cdot ik}), \hat{A}_\alpha)\hat{E}_k + \hat{}(E_i(A_\alpha) + [\Gamma^{\cdot}_{i\cdot}, A_\alpha]), \tag{9}$$

$$\hat{\nabla}_{\hat{A}_\alpha}\hat{E}_i = \tfrac{1}{2}h(\hat{}(R^{\cdot}_{\cdot ik}), \hat{A}_\alpha)\hat{E}_k, \tag{10}$$

$$\hat{\nabla}_{\hat{A}_\alpha}\hat{A}_\beta = \check{\nabla}_{\hat{A}_\alpha}\hat{A}_\beta. \tag{11}$$

Note that formula (11) holds for any pair of vertical vector $\hat{A}$, $\hat{B}$. Now we recall the definition of the O'Neill tensor $\mathcal{A}$. For any pair of vector fields ξ, η on $Z_g(M)$ let $\mathcal{A}_\xi\eta$ be defined by

$$\mathcal{A}_\xi\eta = (\hat{\nabla}_{\xi^{\mathrm{H}}}\eta^{\mathrm{V}})^{\mathrm{H}} + (\hat{\nabla}_{\xi^{\mathrm{H}}}\eta^{\mathrm{H}})^{\mathrm{V}}.$$

Then we have:

$$\mathcal{A}_{\hat{E}_i}\hat{E}_j = -\tfrac{1}{2}\hat{}(R^{\cdot}_{\cdot ij}), \tag{12}$$

$$\mathcal{A}_{\hat{E}_i}\hat{A}_\alpha = \tfrac{1}{2}h(\hat{}(R^{\cdot}_{\cdot ik}), \hat{A}_\alpha)\hat{E}_k, \tag{13}$$

$$\mathcal{A}_{\hat{A}_\alpha}\hat{E}_i = 0, \tag{14}$$

$$\mathcal{A}_{\hat{A}_\alpha}\hat{A}_\beta = 0. \tag{15}$$

3. Twistor isometries

In order to estimate the dimension of the isometry group of the twistor space $Z_g(M)$, we start by proving the following

Proposition 3.1. *Let (M, g) be an oriented $2n$-dimensional Riemannian manifold and $(Z_g(M), h)$ its twistor space endowed with the twistor metric. If the Ricci tensor of (M, g) is parallel and nonpositive, then every isometry of $(Z_g(M), h)$ preserves the horizontal and vertical distributions.*

To prove the proposition, we need the following fact of linear algebra

Lemma 3.1. *Let (E, h) be a Euclidean vector space. Let H be a subspace of E and S be a symmetric bilinear form on E such that*

$$S(u, u) \le 0 \quad \forall u \in H; \qquad S(u, w) = 0 \quad \forall u \in H, w \in H^\perp;$$
$$S(w, w) > 0 \quad \forall w \in H^\perp, w \ne 0,$$

where $\perp$ means the orthogonal with respect to h. If a linear isometry f of (E, h) preserves S, then $f(H) = H$ and $f(H^\perp) = H^\perp$.

Proof. Let $v \in H$, $v \ne 0$ and $f(v) = u + w$, where $u \in H$ and $w \in H^\perp$. We will prove that $w = 0$. Since f is an isometry preserving S and $S(u, w) = 0$ for all $u \in H$, $w \in H^\perp$, we have

$$S(f(H), f(H)^\perp) = S(f(H), f(H^\perp)) = S(H, H^\perp) = 0. \tag{16}$$

Notice that $u \ne 0$, otherwise $f(v) = w \in H^\perp \setminus \{0\}$ and, consequently,

$$0 \ge S(v, v) = S(f(v), f(v)) = S(w, w) > 0.$$

We consider the vector $-(h(w, w)/h(u, u))u + w$, h-orthogonal to $f(v) = u + w$. Therefore, by Eq. (16), we get

$$0 = S\left(u + w, -\frac{h(w, w)}{h(u, u)}u + w\right) = -\frac{h(w, w)}{h(u, u)}S(u, u) + S(w, w).$$

This implies $S(w, w) = 0$ and so $w = 0$. This ends the proof. □

Proof of Proposition 3.1. We compute the Ricci tensor $\hat{S}$ of $(Z_g(M), h)$. Let X, Y be horizontal vector fields on $Z_g(M)$, V, W be vertical vector fields on $Z_g(M)$ and $\{\hat{E}_1, \ldots, \hat{E}_{2n}\}$ a local orthonormal basis of the horizontal distribution. Let us denote by S and $\check{S}$ the Ricci tensors of M and of the fibre of $Z_g(M)$, respectively. By the O'Neill formulas for the curvature (see e.g. [1,4]) we get

$$\hat{S}(X, Y) = S(X, Y) - 2h(\mathcal{A}_X \hat{E}_i, \mathcal{A}_Y \hat{E}_i), \qquad \hat{S}(X, W) = h((\hat{\nabla}_{\hat{E}_i} \mathcal{A}_X)_{\hat{E}_i} X, W),$$
$$\hat{S}(V, W) = \check{S}(V, W) + h(\mathcal{A}_{\hat{E}_i} V, \mathcal{A}_{\hat{E}_i} W).$$

Since S is nonpositive, by the last expressions, $\hat{S}$ is also nonpositive on the horizontal distribution and positive on the vertical one. Now we will prove that $\hat{S}(X, W) = 0$. We may assume that $X = \hat{E}_j$. We have

$$\begin{aligned}(\hat{\nabla}_{\hat{E}_i} \mathcal{A})_{\hat{E}_i} \hat{E}_j &= \hat{\nabla}_{\hat{E}_i} \mathcal{A}_{\hat{E}_i} \hat{E}_j - \mathcal{A}_{\hat{\nabla}_{\hat{E}_i} \hat{E}_i} \hat{E}_j - \mathcal{A}_{\hat{E}_i} \hat{\nabla}_{\hat{E}_i} \hat{E}_j \\ &= -\tfrac{1}{4} h(\hat{}R^{\cdot}_{\cdot ik}, \hat{}R^{\cdot}_{\cdot ij})\hat{E}_k - \tfrac{1}{2}\hat{}(E_i(R^{\cdot}_{\cdot ij}) + [\Gamma^{\cdot}_{i\cdot}, R^{\cdot}_{\cdot ij}]) + \tfrac{1}{2}\Gamma^k_{ii}(\hat{}R^{\cdot}_{\cdot kj}) \\ &\quad + \tfrac{1}{4} h(\hat{}(R^{\cdot}_{\cdot ij}), \hat{}(R^{\cdot}_{\cdot ik}))\hat{E}_k + \tfrac{1}{2}\Gamma^k_{ij}\hat{}(R^{\cdot}_{\cdot ik}).\end{aligned}$$

Therefore,

$$\begin{aligned}\hat{S}(\hat{E}_j, W) &= \tfrac{1}{2} h(\hat{}(-E_i(R^l_{mij}) - \Gamma^l_{ir} R^r_{mij} + \Gamma^r_{im} R^l_{rij}), W) \\ &\quad + \tfrac{1}{2} h(\hat{}(\Gamma^k_{ii} R^l_{mkj} + \Gamma^r_{im} R^l_{rij} + \Gamma^k_{ij} R^l_{mik}), W) \\ &= -\tfrac{1}{2} h(\hat{}((\nabla_{E_i} R)(E_l, E_m, E_i, E_j)), W).\end{aligned}$$

Since the Ricci tensor S of M is parallel, we get

$$(\nabla_{E_i} R)(E_l, E_m, E_i, E_j) = 0,$$

and so $\hat{S}(\hat{E}_j, W) = 0$. By applying Lemma 3.1 to each tangent space to $Z_g(M)$, we have that every isometry of $(Z_g(M), h)$ preserves the horizontal and vertical distributions. This ends the proof. □

Now we can give a characterisation of the infinitesimal isometries of $(Z_g(M), h)$.

Corollary 3.1. *Let ξ be any Killing vector field on the twistor space $(Z_g(M), h)$ over an oriented $2n$-dimensional Riemannian manifold (M, g) with nonpositive and parallel Ricci tensor. Then there exist a Killing vector field X on (M, g) and a vertical Killing vector field V on $(Z_g(M), h)$ such that*

$$\xi = X\hat{} + V,$$

$X\hat{}$ being the horizontal lift of X on $Z_g(M)$.

Proof. Let φ_t be the local one-parameter group of isometries of $(Z_g(M), h)$ induced by ξ. By Proposition 3.1 φ_t preserves the horizontal and vertical distributions. Therefore,

$$\varphi_t(x, P) = (\theta_t(x), \psi_t(x, P)),$$

where θ_t is a local one-parameter group of isometries on (M, g). Let X be the Killing vector field on M induced by θ_t and $X\hat{}$ be its horizontal lift on $Z_g(M)$. Then $X\hat{}$ is a Killing vector field and $V = \xi - X\hat{}$ is the vertical component of ξ. □

For the infinitesimal isometries of $(Z_g(M), h, \mathbb{J})$, that are holomorphic with respect to the almost complex structure $\mathbb{J}$ on $Z_g(M)$, we have the following lemma.

Lemma 3.2. *Let $(Z_g(M), h, \mathbb{J})$ be the twistor space over an oriented $2n$-dimensional Riemannian manifold (M, g). If $n > 2$, then there are no nontrivial holomorphic vertical Killing vector fields on $(Z_g(M), h, \mathbb{J})$.*

Proof. Let V be a vertical Killing vector field. V is infinitesimally holomorphic on $Z_g(M)$ if and only if $L_V\mathbb{J} = 0$, that is equivalent to the following system

$$[V, \mathbb{J}\hat{E}_i] - \mathbb{J}[V, \hat{E}_i] = 0, \qquad [V, \mathbb{J}\hat{A}_\alpha] - \mathbb{J}[V, \hat{A}_\alpha] = 0, \tag{17}$$

where $i = 1, \dots, 2n$, $\alpha = 1, \dots, n^2 - n$. The first of Eq. (17) gives

$$[V, P_i^l \hat{E}_l] - \mathbb{J}[V, \hat{E}_i] = 0. \tag{18}$$

Taking the horizontal component of Eq. (18) and recalling formula (6), we get $V(P_i^l) = 0$, for $i, l = 1, \dots, 2n$. Since $V = [A, P](\partial/\partial P)$, we have $[A, P] = 0$ and, consequently, $A = 0$. □

We consider now the groups $\mathrm{Iso}(Z_g(M), h)$ of isometries of $(Z_g(M), h)$ and $\mathrm{Iso}_{\mathrm{hol}}(Z_g(M), h, \mathbb{J})$ of holomorphic isometries of $(Z_g(M), h, \mathbb{J})$. We can prove the following theorem.

Theorem 3.1. *Let (M, g) be a compact oriented $2n$-dimensional Riemannian manifold with nonpositive and parallel Ricci tensor.*

Let $(Z_g(M), h, \mathbb{J})$ be the twistor space of (M, g). Then, for $n > 2$, we have

$$\dim\mathrm{Iso}(Z_g(M), h) \leq n(2n + 1), \tag{19}$$

$$\dim\mathrm{Iso}_{\mathrm{hol}}(Z_g(M), h, \mathbb{J}) \leq 2n. \tag{20}$$

Equalities hold if and only if M is a torus.

Proof. A theorem of Bochner (see e.g. [8]) says that the isometry group $\mathrm{Iso}(M, g)$ of (M, g) has dimension less than or equal to $\dim M$. In order to prove (19), by Corollary 3.1, we are going to estimate the dimension of the space of vertical Killing vector fields on $(Z_g(M), h)$. Let V be a vertical Killing vector field on $Z_g(M)$. We may write

$$V = [A, P]\frac{\partial}{\partial P},$$

where $A = A(x) \in \mathfrak{so}(2n)$, for all $x = r(P)$. As we already said, we can think V as an element of $\mathfrak{so}(T_{r(P)}M, g_{r(P)})$ such that $P \circ V = -V \circ P$. The vector field V is Killing if and only if, for any pair of vector fields ξ, η on $Z_g(M)$, we have

$$h(\hat{\nabla}_\xi V, \eta) + h(\hat{\nabla}_\eta V, \xi) = 0. \tag{21}$$

Take $\xi = \hat{E}_i$, for $i = 1, \ldots, 2n$, and $\eta = \hat{A}_\alpha$, for $\alpha = 1, \ldots, n^2 - n$. Then, by formula (11), $\hat{\nabla}_\eta V$ is vertical and so $h(\hat{\nabla}_{\hat{E}_i} V, \hat{A}_\alpha) = 0$.

On the other hand,

$$h(\hat{\nabla}_{\hat{E}_i} V, \hat{A}_\alpha) = h(\hat{}(E_i(A)+[\Gamma_{i\cdot}^{\cdot}, A]), \hat{A}_\alpha) = h(\hat{}(E_i(A_m^l) + \Gamma_{ik}^l A_m^k - \Gamma_{im}^k A_k^l), \hat{A}_\alpha)$$
$$= h(\hat{}(\nabla_{E_i} A), \hat{A}_\alpha) = 0,$$

for every $\hat{A}_\alpha$. Therefore, $\hat{}(\nabla_{E_i} A) = 0$ for $i = 1, \ldots, 2n$ and this implies $\nabla_{E_i} A = 0$, since $\hat{}$ is an isomorphism for $n > 2$, as we recalled.

Take now $\xi = \hat{E}_i$, $\eta = \hat{E}_j$. Formula (9) yields

$$h(\hat{\nabla}_{\hat{E}_i} V, \hat{E}_j) + h(\hat{\nabla}_{\hat{E}_j} V, \hat{E}_i) = \tfrac{1}{2} h(\hat{}(R_{\cdot ij}^{\cdot}), \hat{A}) + \tfrac{1}{2} h(\hat{}(R_{\cdot ji}^{\cdot}), \hat{A}) = 0,$$

and, in this case, formula (21) gives no conditions on $A = A(x)$.

Finally, if $\xi = \hat{A}_\alpha$ and $\eta = \hat{A}_\beta$, then Eq. (21) reduces to the Killing condition on the fibre. Therefore, we may identify the space of vertical Killing vector fields on $(Z_g(M), h)$ to the space of parallel 2-forms on M. Hence, Eq. (19) is proved.

By Lemma 3.2, there are no nontrivial holomorphic Killing vector fields along the fibres of $Z_g(M)$. Hence, the estimate (20) follows at once.

The equality in (19) holds if and only if $\dim \mathrm{Iso}(M, g) = 2n$ and, by the theorem of Bochner, this is equivalent to say that M is a torus.

If the equality in Eq. (20) holds, then $\dim \mathrm{Iso}(M, g) = 2n$ and M is a torus. Vice versa, if M is a torus, then any translation lifts to a holomorphic transformation of $Z_g(M)$ and so $\dim \mathrm{Iso}_{\mathrm{hol}}(Z_g(M), h, \mathbb{J}) = 2n$. □

By the vanishing theorem of Bochner for holomorphic vector fields on almost Kähler manifolds and the previous theorem, we get the following corollary.

Corollary 3.2. *Let (M, g) be a compact almost Kähler manifold with nonpositive and parallel Ricci tensor. If the Ricci tensor is negative, then there are no nontrivial holomorphic Killing vector fields on the twistor space $(Z_g(M), h, \mathbb{J})$ and the group of the holomorphic isometries is finite*.

For further reading see [12].

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